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Optimal Control Law of Inclined Inverted Pendulum

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Abstract: *This paper focuses on the derivation of the optimal control law of an inclined inverted pendulum. The condition of complete controllability requires that there be a control that can take any nonzero state $x(t_o)$ at time t_o to the zero state at some time T. For both time-invariant and time-varying systems, T can be taken as close to t_o as desired. Whenever this type of situation arises due to the choice of the control scheme that is adopted then, there will be an increase in control energy required to effect the state transition. The effect of this on the performance index requirement and its minimization with respect to an inclined inverted pendulum (IP) analyzed using Hamilton-Jacobi optimization technique.*

Keyword: *Inclined inverted pendulum; optimal control law; state-space description; Hamilton-Jacobi minimization, performance index.*

1. INTRODUCTION

The importance of IP in several industrial and space exploration fields has been continuously growing fairly recently with the increasing need for the development of control systems able to improve continuously the integrity of complex systems.

Stability, transient response, bandwidth, disturbance rejection, steady state error, and robustness to plant variations or uncertainties have limited the use of control systems in industrial applications [1]. Most often control systems are composed of parts motivated by the need to eliminate or reduce the effect of unwanted disturbances that perturb their outputs. The suppression of unwanted disturbances can be achieved by a control technique which transits the plant from a nonzero state to the zero state [1]. In classical control, this is achieved by using *feedback* of the output and its derivatives to generate a control or command signal.

Significant progress has been made towards developing design techniques for improving the performance of control systems. Pole-placement method, where the desired performance is indirectly achieved through the location of closed-loop poles, has reasonable potential for solving control stability and other related problems. However, the same method is found to be dominated by over-abundance of design parameters for multi-input, multi-output systems [2]. In spite of the successful design exploitation of polepla-cement method, it is difficult to figure out how to determine all the design parameters, because only a limited number of them are likely to be found from the closed-loop pole locations [2].

However, there is a clear need for a high sensitivity and reliability control design technique that can have readily computable solutions, applicable to nonlinear systems operating on a small signal basis, good robustness properties such as gain margin and phase margin, while still being deployed for optimality of a quadratic index [2]. Such a system will be utilized to directly address the desired performance objectives, while minimizing the control energy. This is the area of influence of linear optimal control (LOC). LOC design is primarily directed towards computing the optimal control law.

LOC allows for direct formulation of the performance objectives of a control system [2], [3]. Owing to its unique potential which allows computational procedures to be adopted for use in solving nonlinear optimal problems, LOC can offer an obvious solution in the development of performance monitoring system, to provide assurance of performance integrity for ageing control systems.

The analytical design approach can allow the entire control system to be characterized by a number of simplified control features, enabling performance curves of the associated control systems to be plotted. Performance predictions, supported by experiments, are used to identify the most promising robustness property in each of the control features. In this way, stability, bandwidth, and any one of the desirable constraints associated with classical control would produce the best possible control system for a given set of performance objectives. When performance curves cannot sufficiently describe the robustness properties,

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dynamic finite element modeling is explored and model predictions are also validated by experiment.

The efficiency or effectiveness of LOC, which is a special sort of optimal control, is critically dependent on the linearity of the plant to be controlled and the controller, the device that generates the optimal control. The technique based on quadratic performance indices is one the most promising candidates that can be used to realize linear controllers [1].

The remaining content of this paper is organized in the following way. The general concept of control system performance index and constrained minimization is presented in section 2. In section 3, the idea in section is tailored towards inclined (IP). A brief discussion of the obtained result is presented in section 4. Final conclusion is drawn in section 5.

2. PERFORMANCE INDEX DEFINITION

In LOC, a time-variant controller with a known transfer function description is widely regarded as capable of significantly reducing the effect of external disturbances on a plant [4]. This is usually inserted between the plant output and plant input. Effective controllers can be considered those which combine good sensitivity to errors, preferably with the capability of generating appropriate command signal across the time-variant plant input. [1].

However, the availability of plant state is an important consideration in design analysis based on optimal control. A continuous update of state integrity will enable better informed performance [1], [5], [6]. When this is not the case, a state estimator driven by both the plant input and output is constructed as shown in Figure 1 [1].

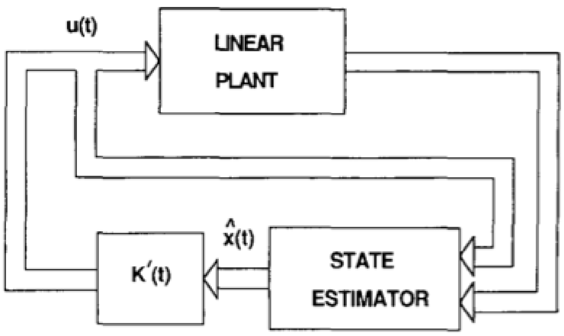

Figure 1 State estimate feedback arrangement [1]

A continuous update of component integrity will enable better informed, targeted inspections and outage maintenance providing increased power generation availability.

A linear, time-varying plant with state-space description in (1) is considered

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad (1)$$

The linear control law that will allow for the design of a full-state feedback regulator for the linear plant described in (1) is given in (2) [1], [2].

$$u(t) = -K(t)x(t) \quad (2)$$

The resultant closed-loop control system would also be linear.

Assuming $R(t)$ is a square, symmetric, positive-definite matrix, then, the control energy C_e needed to generate the control input $u(t)$ is expressed as

$$C_e = u^{'}(t)R(t)u(t) \quad (3)$$

Expression in (3) is a quadratic functions of the elements of $u(t)$.

In the design of controllers, the condition of complete controllability requires that there be a control that can take any nonzero state $x(t_o)$ at time t_o to the zero state at some time T. For both time-invariant and time-varying systems, T can be taken as close to t_o as desired [1], [2]. Whenever this type of situation arises due to the choice of the control scheme that is adopted then, there will be an increase in control energy required to effect the state transition. Therefore, T cannot be arbitrarily close to t_o without exceeding the upper bounds on the variables of the system.

There is the need to keep some measure of control input bounded during the course of a control action such that

$$C_e = \int_{t_o}^{T} u^{'}(t)R(t)u(t)dt \quad (4)$$

Also, $K(T)$ may not necessarily be implementable.

Also some energy is required to be expended to handle the transient response of the system. The quadratic form of the maximum value T_e of this energy referred to as transient energy which, quickly decays to zero is

$$T_e = x^{'}(t)Q(t)x(t) \quad (5)$$

$Q(t)$ is a *square*, symmetric matrix called the state weighting matrix.

Instead of the transient response to actually achieve the zero state, the requirement can be merely that the state as measured by some norm should become small. Therefore, it is necessary to have $\|x(t)\|$ small for any t in the interval over which control is being exercised [2]. This fact can be expressed by

$$T_e = \int_{t_o}^{T} x^{'}(t)Q(t)x(t)dt \quad (6)$$

By adding (4) and (6), the objective function or performance index is written as

$$J(t_o, T) = \int_{t_o}^{T}\left(u^{'}(t)R(t)u(t) + x^{'}(t)Q(t)x(t)\right)dt \quad (7)$$

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Expression (7) is generally referred to as linear quadratic cost function and is one of the several cost functions that exits. However, piecewise quadratic cost functions is well-studied and have found extensive application in the stability analysis of piecewise linear systems and performance analysis and optimal control [7]. In this method, semi-definite programming based on the Bellman inequality is employed in the computation of the lower bounds of optimal control cost function.

In the study of the concept of optimal control problems with control constraints proposed in [8], first order optimality conditions and the discretization of the state and adjoint equations can help in the discretization of the control.

Consequently, the optimal control problem involves solving for the feedback gain matrix, $K(t)$, such that the scalar performance index, $J(t_o, T)$, expressed in. (7) is minimized. However, the minimization must be implemented in such a way that the state-vector, $x(t)$, is the solution of the plant's state-equation (1). Equation (1) is called a constraint (because in its absence, $x(t)$ would be free to assume any value), and the resulting minimization is said to be a constrained minimization [2]. Hence, the problem is to seek a linear control law, $K(t)$, which minimizes $J(t_o, T)$ subject to the dynamic constraint (1). The idea presented in this section will be used to compute optimal control law of an inclined IP.

3. FORMULATION OF MINIMIZATION PROBLEM

Equations (8) and (9) are second-order differential equations which constitute the nonlinear description of an inclined IP shown in Figure 2 [9].

$$\frac{d^2x}{dt^2} = \frac{\frac{1}{4}\left[4f_a\left(1+\frac{1}{4f_a}\right)^{\frac{1}{M_p}} - 3g\varphi\right]}{\left(\frac{M_s}{M_p}+\frac{1}{4}\right)} = \frac{\frac{1}{4}\left[4\frac{f_a}{M_p} - 3g\varphi\right]}{\left(\frac{M_s}{M_p}+\frac{1}{4}\right)} = \frac{\frac{f_a}{M_p} - \frac{3g\varphi}{4}}{\frac{M_s}{M_p}+\frac{1}{4}} \quad (10)$$

In the IP consideration, if x and φ are the outputs, then the state variable representation of the linearized model can be obtained by assuming the following states.

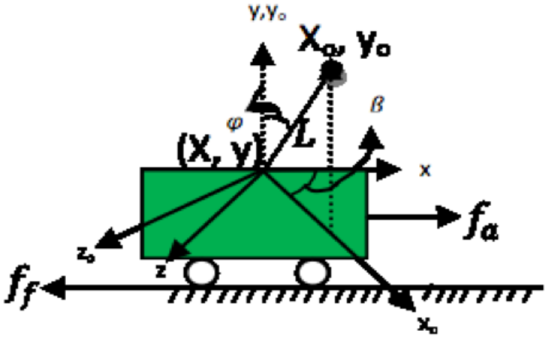

Figure 2 Inclined Inverted Pendulum System [9]

$$\frac{d^2\varphi}{dt^2} = \frac{\frac{3}{4}\left[\left(1+\frac{M_s}{M_p}\right)g\,\varphi \;-\; \frac{f_a}{M_p}\right]}{L\left[\;1+\frac{M_s}{M_p} - \frac{3}{4}\right]} = \frac{\frac{3}{4}\left[\left(1+\frac{M_s}{M_p}\right)g\,\varphi \;-\; \frac{f_a}{M_p}\right]}{L\left[\frac{1}{4}+\frac{M_s}{M_p}\;\right]} = \frac{\left(1+\frac{M_s}{M_p}\right)g\,\varphi \;-\; \frac{f_a}{M_p}}{\frac{4}{3}L\left[\frac{1}{4}+\frac{M_s}{M_p}\;\right]} \quad (11)$$

$$\left.\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ x_3 &= \varphi \\ x_4 &= \dot{\varphi} \end{aligned}\right\} \quad (12)$$

Using (10) and (11), the states expressed in (12) can be rewritten as first-order differentials equations. Thus we have

$$\frac{dx_1}{dt} = \frac{dx}{dt} = x_2 \quad (13)$$

$$\frac{dx_2}{dt} = \frac{d^2x}{dt^2} = \left(\frac{4}{4M_s+M_p}\right)f_a - \left(\frac{3gM_P}{4M_s+M_p}\right)x_3 \quad (14)$$

$$\frac{dx_3}{dt} = \frac{d\varphi}{dt} = x_4 \quad (15)$$

$$\frac{dx_4}{dt} = \frac{d^2\varphi}{dt^2} = \frac{3g(M_s+M_p)}{L(4M_s+M_p)}x_3 - \left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right)f_a \quad (16)$$

Eqn. (13) through (16) can be expressed in state variable form as

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$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\left(\frac{3gM_P}{4M_s+M_p}\right) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{3g(M_s+M_p)}{L(4M_s+M_p)} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \left(\frac{4}{4M_s+M_p}\right) \\ 0 \\ -\left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right) \end{bmatrix} \quad (17)$$

However, when A, B, C, and D are constant, the generalized state variable representation is

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (18)$$

$$y = Cx(t) + Du(t) \quad (19)$$

Where A is the coefficient matrix, B is the driving matrix, C is the output matrix, D is the transmission matrix. y is the output. Since x and φ constitute the output of the system, they will be represented by y_1 and y_2. Comparing eqns. (17) and (18), we can write that

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\left(\frac{3gM_P}{4M_s+M_p}\right) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{3g(M_s+M_p)}{L(4M_s+M_p)} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \left(\frac{4}{4M_s+M_p}\right) \\ 0 \\ -\left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right) \end{bmatrix}$$

$$C_1 = [1 \quad 0 \quad 0 \quad 0]$$
$$C_2 = [0 \quad 0 \quad 0 \quad 1] \quad (20)$$

It is necessary to seek a control u which makes the Inclined IP system, $v(x(t), u(t), t)$ expressed by (1) to optimally trace a path $x(t)$ which minimizes a performance index expressed as

$$J = \int_{t_i}^{t_f} s(x(t), u(t), t)dt \quad (21)$$

Assuming the performance index has a minimum value $g(x, t)$ defined over the time interval t_i (initial time) to t_f (final time) such that

$$g(x, t) = \min_u \int_{t_i}^{t_f} s(x, t)dt \quad (22)$$

Within defined time interval the following is true $g(x, t_i) = g(x(0))$ and $g(x, t_f) = 0$. If (22) is considered to be a Lyapunov function, its derivative can be equated to the differentials obtained through the application of chain rule to the same eqn. Therefore, we get

$$\frac{\partial g(x,t)}{\partial t} + \left(\frac{\partial g(x,t)}{\partial t}\right)^T v(x, u) = -s(x(t), u(t), t) \quad (23)$$

$$\frac{\partial g(x,t)}{\partial t} = -\min_u \left(\left(\frac{\partial g(x,t)}{\partial t}\right)^T v(x, u) + sxt,ut,t\right) \quad (24)$$

Eqn. (15) is referred to as Hamilton-Jacobi equation, where the superscript T means transpose of the matrix in question [1]. Let the performance index of (21) be the quadratic type. We can write (21) as

$$J = \int_{t_i}^{t_f} (x^T N x + u^T R u)dt \quad (25)$$

Substituting (18) and (25) in (24) we have

$$\frac{\partial g(x,t)}{\partial t} = -\min_u \left(\left(\frac{\partial g(x,t)}{\partial t}\right)^T (Ax + Bu) + xTNx+uTRu\right) \quad (26)$$

Where $A(t)$, $B(t)$, $N(t)$ *and* $R(t)$ are continuously differentiable functions. Also, $N(t)$ is a symmetric semi-definite matrix *and* $R(t)$ is a symmetric definite matrix. Let us consider a square symmetric matrix, (Ricatti matrix), M such that

$$g(x, t) = x^T M x \quad (27)$$

From (27) we can write

$$\frac{\partial g(x,t)}{\partial x} = 2Mx \quad (28)$$

$$\left[\frac{\partial g(x,t)}{\partial x}\right]^T = 2Mx^T \quad (29)$$

$$\frac{\partial g(x,t)}{\partial t} = x^T \left(\frac{\partial M}{\partial t}\right) x \quad (30)$$

Therefore, (26) becomes

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$$\frac{\partial g(x,t)}{\partial t} = -\min_u (2Mx^T(Ax+Bu) + x^T Nx + uTRu \quad (30)$$

Differentiating (30) with respect to u and equating to zero, we have

$$2MBx^T + 2Ru^T = 0 \quad (31)$$

The optimal value of the control u^o is obtained as

$$u^o = f_a = -R^{-1}MB^T x = -kx \quad (32)$$

Where

$$k = R^{-1}MB^T \quad (33)$$

Substituting eqns. (26) and (29) in (22), we have

$$x^T\left(\frac{\partial M}{\partial t}\right)x = -x(N + 2MA - 2MBR^{-1}B^T M + MBR-1BTMxT \quad (34)$$

$$x^T\left(\frac{\partial M}{\partial t}\right)x = -x(N + 2MA - MBR^{-1}B^T M)x^T \quad (35)$$

Eqn. (35) simplifies to

$$\frac{\partial M}{\partial t} = \dot{M} = MBR^{-1}B^T M - A^T M - MA - N \quad (36)$$

Eqn. (36) is the matrix Ricatti equations for the Inclined IP system. The solutions converge to constant values as integration is performed in the reverse direction [10]. If t_f is further away from t_i, then matrix Ricatti equations reduce to

$$A^T M + MA + N - MBR^{-1}B^T M = 0 \quad (37)$$

Equation (37) represents a group of simultaneous equations.

Let us consider an inclined IP system which will minimize a performance, J index such that

$$J = \int_0^\infty (x_1^2 + x_2^2 + u_1^2 + u_2^2)dt \quad (38)$$

$$MA = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \quad 0 & \\ 0 & 0 & -\left(\frac{3gM_P}{4M_s+M_p}\right) & 0 \\ 0 & 0 & 0 \quad 1 & \\ 0 & 0 & \left(\frac{3g(M_s+M_p)}{L(4M_s+M_p)}\right) & 0 \end{bmatrix} \quad (39)$$

$$MA = \begin{bmatrix} 0 & M_{11} & \left(\left(\frac{3g(M_s+M_p)}{L(4M_s+M_p)}\right)M_{14} - \left(\frac{3gM_P}{4M_s+M_p}\right)M_{12}\right) & M_{13} \\ 0 & M_{21} & \left(\left(\frac{3g(M_s+M_p)}{L(4M_s+M_p)}\right)M_{24} - \left(\frac{3gM_P}{4M_s+M_p}\right)M_{22}\right) & M_{23} \\ 0 & M_{31} & \left(\left(\frac{3g(M_s+M_p)}{L(4M_s+M_p)}\right)M_{34} - \left(\frac{3gM_P}{4M_s+M_p}\right)M_{32}\right) & M_{33} \\ 0 & M_{41} & \left(\left(\frac{3g(M_s+M_p)}{L(4M_s+M_p)}\right)M_{44} - \left(\frac{3gM_P}{4M_s+M_p}\right)M_{42}\right) & M_{43} \end{bmatrix} \quad (40)$$

$$A^T M = \begin{bmatrix} 0 & 0 & 0 \quad 0 & \\ 1 & 0 & 0 \quad 0 & \\ 0 & -\left(\frac{3gM_P}{4M_s+M_p}\right) & 0 & \left(\frac{3g(M_s+M_p)}{L(4M_s+M_p)}\right) \\ 0 & 0 & 1 \quad 0 & \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \quad (41)$$

$$A^T M = \begin{bmatrix} 3\omega^2 M_{21} & 3\omega^2 M_{22} & 3\omega^2 M_{23} & 3\omega^2 M_{24} \\ \left(M_{11} - {}^{2\omega}/_{R_o} M_{41}\right) & \left(M_{12} - {}^{2\omega}/_{R_o} M_{42}\right) & \left(M_{13} - {}^{2\omega}/_{R_o} M_{43}\right) & \left(M_{14} - {}^{2\omega}/_{R_o} M_{44}\right) \\ 0 & 0 & 0 & 0 \\ (2R_o\omega M_{21} + M_{31}) & (2R_o\omega M_{22} + M_{32}) & (2R_o\omega M_{23} + M_{33}) & (2R_o\omega M_{24} + M_{34}) \end{bmatrix} \quad (42)$$

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$$N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{43}$$

$$MBR^{-1}B^TM =$$

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \begin{bmatrix} 0 \\ \left(\frac{4}{4M_s+M_p}\right) \\ 0 \\ -\left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right) \end{bmatrix} R^{-1} \begin{bmatrix} 0 & \left(\frac{4}{4M_s+M_p}\right) & 0 & -\left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right) \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \tag{44}$$

$$MBR^{-1}B^TM =$$

$$\begin{bmatrix}
\left[\left(\frac{4}{4M_s+M_p}\right)M_{12} - \left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right)M_{14}\right]\left[\left(\frac{4}{4M_s+M_p}\right)M_{21} - \left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right)M_{41}\right] & \left[\left(\frac{4}{4M_s+M_p}\right)M_{12} - \left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right)M_{14}\right]\left[\left(\frac{4}{4M_s+M_p}\right)M_{22} - \left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right)M_{42}\right] & \frac{1}{Rm^2}\left(M_{12}M_{23} + \frac{M_{14}M_{43}}{R_o^2}\right) & \frac{1}{Rm^2}\left(M_{12}M_{24} + \frac{M_{14}M_{44}}{R_o^2}\right) \\
\left[\left(\frac{4}{4M_s+M_p}\right)M_{22} - \left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right)M_{24}\right]\left[\left(\frac{4}{4M_s+M_p}\right)M_{21} - \left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right)M_{41}\right] & \left[\left(\frac{4}{4M_s+M_p}\right)M_{22} - \left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right)M_{24}\right]\left[\left(\frac{4}{4M_s+M_p}\right)M_{22} - \left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right)M_{42}\right] & \frac{1}{Rm^2}\left(M_{22}M_{23} + \frac{M_{24}M_{43}}{R_o^2}\right) & \frac{1}{Rm^2}\left(M_{22}M_{24} + \frac{M_{24}M_{44}}{R_o^2}\right) \\
\left[\left(\frac{4}{4M_s+M_p}\right)M_{32} - \left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right)M_{34}\right]\left[\left(\frac{4}{4M_s+M_p}\right)M_{21} - \left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right)M_{41}\right] & \left[\left(\frac{4}{4M_s+M_p}\right)M_{32} - \left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right)M_{34}\right]\left[\left(\frac{4}{4M_s+M_p}\right)M_{22} - \left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right)M_{42}\right] & \frac{1}{Rm^2}\left(M_{23}^2 + \frac{M_{34}^2}{R_o^2}\right) & \frac{1}{Rm^2}\left(M_{32}M_{24} + \frac{M_{34}M_{44}}{R_o^2}\right) \\
\left[\left(\frac{4}{4M_s+M_p}\right)M_{42} - \left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right)M_{44}\right]\left[\left(\frac{4}{4M_s+M_p}\right)M_{21} - \left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right)M_{41}\right] & \left[\left(\frac{4}{4M_s+M_p}\right)M_{42} - \left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right)M_{44}\right]\left[\left(\frac{4}{4M_s+M_p}\right)M_{22} - \left(\frac{3(M_s+M_p)}{L(4M_s+M_p)}\right)M_{42}\right] & \frac{1}{Rm^2}\left(M_{42}M_{23} + \frac{M_{43}M_{44}}{R_o^2}\right) & \frac{1}{Rm^2}\left(M_{24}^2 + \frac{M_{44}^2}{R_o^2}\right)
\end{bmatrix} \tag{45}$$

When the terms expressed in (40) through (45) are substituted in (37), the matrix M can be computed. Hence the optimal control law of (32) is realized.

4. DISCUSSION

In this analysis, the characteristic of an optimal control law of inclined IP based on a quadratic performance index is a linear function of the state variables, which implies that there is availability and feedback of all state variables. In terms of implementation, state observer will be attractive in the estimation of un-measurable state variables.

Although, Runge-Kutta approach is one of the effective methods that can be explored in seeking solution to a first-order nonlinear equation such as Ricatti equation [2], the analysis presented has shown that the computation of optimal control law can be obtained by systematic matrix simplification of Riccati equation.

Finally, in agreement to the idea in [2], the optimal control procedure using full-state feedback consists of specifying an objective function by suitably selecting the performance and control cost weighting matrices, $Q(t)$ and $R(t)$, and solving the Riccati equation subject to the terminal condition, in order to determine the full-state feedback matrix, $K(t)$. In most cases, rather than solving the general time-varying optimal control problem, certain simplifications can be made which result in an easier problem, as seen in the following sections.

5. CONCLUSION

In this work, an optimization problem requiring the minimization of performance index of IP has been considered. The overall analysis that employs a partial differential equation, the Hamilton–Jacobi equation, satisfied by the optimal performance index under differentiability and continuity assumptions has been considered. Moreover, it has been shown that since the solution to the Hamilton-Jacobi equation has certain differentiability properties, then this solution is the desired performance index.

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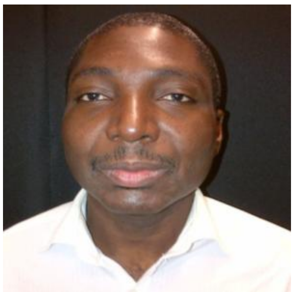
Authors Biography



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